Individual power operations: formulas

Recall from lecture 1: ℓ prime

Eilenberg Maclane Spectrum

$A(\ell) :=$ ring of classical stable operations on $H^*(-, \mathbb{F}_\ell)$ $= H\mathbb{F}_\ell^*(H\mathbb{F}_\ell)$

Yoneda

Axioms for $A(2)$:

1) $A(2)$ is generated by Steenrod squares Sq^i $i = 0, 1, 2, \dots$
2) $Sq^0 = id$
3) $x \in H^n \Rightarrow Sq^n(x) = x^2$
4) $x \in H^n \Rightarrow Sq^i(x) = 0$ for $i > n$
5) Cartan formula: $Sq^k(xy) = \sum_{i+j=k} Sq^i(x)\, Sq^j(y)$
6) $Sq^1 = \beta$ Bockstein = coboundary of $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$
7) Adem relations: $0 < a < 2b \Rightarrow Sq^a Sq^b = \sum_{j=0}^{\lfloor a/2 \rfloor} \binom{b-1-j}{a-2j} Sq^{a+b-j} Sq^j$

Today:

$\ell \neq \operatorname{char} k$ prime

Will define

$$P^i : H^{p,q}(-,\mathbb{Z}/\ell) \longrightarrow H^{p+2i(\ell-1),\, q+i(\ell-1)}(-,\mathbb{Z}/\ell) \quad (= Sq^{2i} \ \ell=2)$$

$$B^i : H^{p,q}(-,\mathbb{Z}/\ell) \longrightarrow H^{p+2i(\ell-1)+1,\, q+i(\ell-1)}(-,\mathbb{Z}/\ell) \quad (= Sq^{2i+1} \ \ell=2)$$

and prove analogs of 2), 3), 4), 5), 6)

and define the total power operation

$$R : H^{*,*}(-,\mathbb{Z}/\ell) \rightsquigarrow \tilde{H}^{*,*}(-,\mathbb{Z}/\ell)[[c,d^{\pm 1}]] \big/ (c^2 = \tau d + \rho c)$$

$\tau = \rho = 0$ for $\ell \neq 2$

$$R(u) := \sum_i B^{i-1}(u)\, c d^i + P^i(u)\, d^{-i}$$

$\rightsquigarrow$ needed to prove Adem relations (next week)

Recall : $P_\ell : K_n \wedge (BS_\ell)_+ \longrightarrow K_{n\ell}$ $\qquad K_n = K_{n,\mathbb{Z}/\ell}$

$\updownarrow$ $\qquad$ Eilenberg - Maclane space

$P_\ell : \tilde{H}^{2n,n}(-,\mathbb{Z}/\ell) \longrightarrow \tilde{H}^{2n\ell,n\ell}(- \wedge (BS_\ell)_+, \mathbb{Z}/\ell)$

Thm 6.16 : $\mathcal{F}_\bullet$ pt simpl sheaf

$$\tilde{H}^{*,*}(\mathcal{F}_\bullet \wedge (BS_\ell)_+, \mathbb{Z}/\ell) = \tilde{H}^{*,*}(\mathcal{F}_\bullet, \mathbb{Z}/\ell)[[c,d]] / c^2 = \tau d + \rho c$$

$|c| = (2\ell-3, \ell-1)$ $\qquad \tau = \rho = 0$ for $\ell \neq 2$

$|d| = (2\ell-2, \ell-1)$

$u \in \tilde{H}^{2n,n}(\mathcal{F}_\bullet, \mathbb{Z}/\ell) \longrightarrow \tilde{H}^{2n\ell,n\ell}(\mathcal{F}_\bullet \wedge (BS_\ell)_+, \mathbb{Z}/\ell) \ni P_\ell(u)$

$\Rightarrow \quad P_\ell(u) = \sum_{i \geq 0} C_{i+1,n}(u) c d^i + D_i(u) d^i$

$C_i := C_{i,n} : \tilde{H}^{2n,n}(-,\mathbb{Z}/\ell) \to \tilde{H}^{2n+2(n-i)(\ell-1)+1,\, n+(n-i)(\ell-1)}(-,\mathbb{Z}/\ell)$

$D_i := D_{i,n} : \tilde{H}^{2n,n}(-,\mathbb{Z}/\ell) \to \tilde{H}^{2n+2(n-i)(\ell-1),\, n+(n-i)(\ell-1)}(-,\mathbb{Z}/\ell)$

Lemma 9.1 $\quad u \in \tilde{H}^{2n,n}(F_0, \mathbb{Z}/\ell), \qquad v \in \tilde{H}^{2n',n'}(F_0, \mathbb{Z}/\ell)$

$$\left.\begin{aligned} &(1) \quad D_i(u \wedge v) = \sum_{r=0}^{i} D_r(u) \wedge D_{i-r}(v) \\ &(2) \quad C_{i+1}(u \wedge v) = \sum_{r=0}^{i} C_{r+1}(u) \wedge D_{i-r}(v) + D_r(u) \wedge C_{i-r+1}(v) \end{aligned}\right\} \ell \neq 2$$

$$\left.\begin{aligned} &(3) \quad D_i(u \wedge v) = \sum_{r=0}^{i} D_r(u) \wedge D_{i-r}(v) + \tau \sum_{r=0}^{i-1} C_{r+1}(u) \wedge C_{i-r}(v) \\ &(4) \quad C_i(u \wedge v) = \sum_{r=0}^{i} C_{r+1}(u) \wedge D_{i-r}(v) + D_r(u) \wedge C_{i-r+1}(v) + \rho\, C_{r+1}(u) \wedge C_{i-r}(v) \end{aligned}\right\} \ell = 2$$

__Pf:__ Lemma 5.7 $\quad P_\ell(u \wedge v) = \Delta^*(P_\ell(u) \wedge P_\ell(v))$

LHS: $\quad P_\ell(u \wedge v) = \sum_{i \geq 0} C_{i+1}(u \wedge v)\, c\, d^i + D_i(u \wedge v)\, d^i$

RHS: $\quad P_\ell(u) \wedge P_\ell(v) = \left(\sum_{j \geq 0} C_{j+1}(u)\, c\, d^j + D_j(u)\, d^j\right) \wedge \left(\sum_{m \geq 0} C_{m+1}(v)\, c d^m + D_m(v)\, d^m\right)$

$$= \sum_{j,m \geq 0} C_{j+1}(u) \wedge C_{m+1}(v)\, c^2 d^{j+m} + \sum_{j,m \geq 0} \left(C_{j+1}(u) \wedge D_m(v) + D_j(u) \wedge C_{m+1}(v)\right) c d^{m+j}$$

$$+ \sum_{j,m \geq 0} D_j(u) \wedge D_m(v)\, d^{j+m}$$

$\square$

Let $\sigma_T \in \tilde{H}^{2,1}(T, \mathbb{Z}/\ell)$ be the tautological class.

Claim: $P_\ell(\sigma_T) = \sigma_T \wedge d$

Pf: $\mathcal{O} \to \xi_\ell \rightsquigarrow th : Th(\mathcal{O}) \to Th(\xi_\ell)$

$$Th(\mathcal{O}) \cong T \wedge BS_{\ell +}$$

monomorphism of bundles over BS_ℓ

Thom class

Lemma 5.8 $\Rightarrow$ $P_\ell(\sigma_T) = th^*(t_{\xi_\ell})$

Lemma 4.7 $\Rightarrow$ $th^*(t_{\xi_\ell}) = \sigma_T \wedge \underbrace{e(\xi_\ell/\mathcal{O})}_{= d}$ $\square$

Lemma 9.2: $C_{i+1}(u \wedge \sigma_T) = C_i(u) \wedge \sigma_T$

$D_i(u \wedge \sigma_T) = D_{i-1}(u) \wedge \sigma_T$

Pf: $P_\ell(\sigma_T) = \sigma_T \wedge d = \underbrace{D_1(\sigma_T)}_{= \sigma_T} \wedge d \Rightarrow$ $C_i(\sigma_T) = 0 \quad \forall i$

$D_i(\sigma_T) = 0 \quad \forall i \neq 1$

Applying Lemma 9.1, we are done ✓

$\square$

$u \in \tilde{H}^{2n,n}(F_\cdot, \mathbb{Z}/\ell)$ By Prop 2.6 the can be extended

$P^i(u) := D_{n-i}(u)$ $\quad P^i : \tilde{H}^{p,q} \to \tilde{H}^{p+2i(\ell-1),\, q+i(\ell-1)}$

$B^i(u) := C_{n-i}(u)$ $\quad B^i : \tilde{H}^{p,q} \to \tilde{H}^{p+2i(\ell-1)+1,\, q+i(\ell-1)}$

For $\ell = 2$ $\quad Sq^{2i} := P^i \quad Sq^{2i+1} := B^i$

Thm 9.3: $P^i = B^i = 0$ for $i < 0$

Pf: $P^i : \tilde{H}^{2n,n}(-, \mathbb{Z}/\ell) \to \tilde{H}^{2n+2i(\ell-1),\, n+i(\ell-1)}(-, \mathbb{Z}/\ell)$

$$\operatorname{Hom}_{\mathcal{H}_\cdot(k)}(-, K_{n,\mathbb{Z}/\ell}) \longrightarrow \operatorname{Hom}_{\mathcal{H}_\cdot(k)}(-, K_{n+i(\ell-1),\mathbb{Z}/\ell})$$

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$\in \operatorname{Hom}(K_{n,\mathbb{Z}/\ell}, K_{n+i(\ell-1),\mathbb{Z}/\ell})$

$= H^{2(n+i(\ell-1)),\, n+i(\ell-1)}(K_{n,\mathbb{Z}/\ell}, \mathbb{Z}/\ell) = 0$

Prop 3.7 $\quad \tilde{H}^{*,m}(K_{n,A}, B) = 0$

$m < n, \quad n \geq 0$

$\square$

Prop 9.6: (weak Cartan formula) Axiom 5)

$\ell \neq 2$: $P^i(u \wedge v) = \sum_{r=0}^{i} P^r(u) \wedge P^{i-r}(v)$

$$B^i(u \wedge v) = \sum_{r=0}^{i} \left(B^r(u) \wedge P^{i-r}(v) + P^r(u) \wedge B^{i-r}(v) \right)$$

$\ell = 2$: $Sq^{2i}(u \wedge v) = \sum_{r=0}^{r} Sq^{2i}(u) \wedge Sq^{2i-2r}(v) + \tau \sum_{s=0}^{i-1} Sq^{2s+1}(u) \wedge Sq^{2i-2s-1}(v)$

$$Sq^{2i+1}(u \wedge v) = \sum_{r=0}^{i} \left(Sq^{2r+1}(u) \wedge Sq^{2i-2r}(v) + Sq^{2r}(u) \wedge Sq^{2i-2r-1}(v) \right) + \rho \sum_{s=0}^{r-1} Sq^{2s+1}(u) \wedge Sq^{2i-2s-1}(v)$$

Pf: $u \in \tilde{H}^{2n,n}$, $v \in \tilde{H}^{2m,m}$

1^{st} formula

Lemma 9.1 (1)

$$P^i(u \wedge v) = D_{n+m-i}(u \wedge v) = \sum_{s=0}^{m+n-i} D_s(u) \wedge D_{m+n-i-s}(v)$$

$$= \sum_{s=0}^{m+n-i} P^{n-s}(u) \wedge P^{i+s-n}(v) = \sum_{r=0}^{i} P^r(u) \wedge P^{i-r}(v)$$

$r = n-s$ + Thm 9.3

$\square$

The rest is similar

Thm 9.4 $\quad P^0 = \mathrm{Id}$ $\quad$ (Axiom 2)

Pf: $P^0 : \tilde{H}^{2n,n}(-, \mathbb{Z}/\ell) \to \tilde{H}^{2n,n}(-, \mathbb{Z}/\ell)$

$$\mathrm{Hom}_{\mathcal{H}.(k)}(-, K_{n,\mathbb{Z}/\ell}) \to \mathrm{Hom}_{\mathcal{H}.(k)}(-, K_{n,\mathbb{Z}/\ell})$$

$$\in \mathrm{Hom}_{\mathcal{H}.(k)}(K_{n,\mathbb{Z}/\ell}, K_{n,\mathbb{Z}/\ell}) = \tilde{H}^{2n,n}(K_{n,\mathbb{Z}/\ell}, \mathbb{Z}/\ell)$$

Yoneda

$$\| \Leftarrow \text{Prop 3.8}$$

$$\mathrm{Hom}(\mathbb{Z}/\ell, \mathbb{Z}/\ell)$$

$\Rightarrow \quad P^0(u) = a \cdot u \quad$ for some $a \in \mathbb{Z}/\ell$

Lemma 6.17: M line bundle

$$P_\ell(e(M)) = \underbrace{e(M)^\ell}_{D_0(e(M))} + \underbrace{e(M)\, d}_{D_1(e(M))}$$

$M = \omega_{\mathbb{P}^1} \quad \Rightarrow \quad e(M) \in \tilde{H}^{2,1}(\mathbb{P}^1, \mathbb{Z}/\ell)$

and $\quad P^0(e(M)) = D_{1-0}(e(M)) = e(M) \quad \Rightarrow a = 1$ $\quad \square$

Lemma 9.5: $\beta \cdot B^i = 0$ and $\beta P^i = B^i$

Axiom 6): for $\ell = 2$ this reads $\beta Sq^0 = Sq^1$, $\beta = Sq^1$

Pf: Have

a) $\beta(c) = d$, $\beta(d) = 0$

b) $\beta(uv) = \beta(u)v + (-1)^p u\beta(v)$ by (8.1) ← first deg of u

c) $u \in \tilde{H}^{2n,n} \Rightarrow \beta P_\ell(u) = 0$ by Thm 8.4

$$0 \overset{c)}{=} \beta P_\ell(u) = \beta\Big(\sum_{i\geq 0} C_{i+1}(u)\, c\, d^i + D_i(u)\, d^i\Big)$$

$$\overset{b)}{=} \sum_{i\geq 0} \beta(B^{n-i-1}(u))\, c\, d^{i-1} + (-1)^{2n+2(n-i)(\ell-1)+1} B^{n-i-1}(u)\, \underbrace{\beta(c)}_{=d \text{ by a)}}\, d^i + \underbrace{\cancel{(-1)^{2n+2(n-i)(\ell-1)+1+2\ell-3} B^{n-i-1}(u)\, c\, \beta(d^i)}}_{=0 \text{ by a)}}$$

$$+ \beta(P^{n-i}(u))\, d^i + \underbrace{\cancel{(-1)^{2n+2(n-i)(\ell-1)} P^{n-i}(u)\, \beta(d^i)}}_{=0 \text{ by a)}}$$

$\Rightarrow \beta B^{n-i-1}(u) = 0$

$\Rightarrow \beta(P^{n-i}(u)) = B^{n-i}(u)$

□

Lemma 9.7 $u \in \tilde{H}^{2n,n} \Rightarrow P^n(u) = u^\ell$ (Axiom 3)

Pf: Lemma 5.10 $\Rightarrow P_\ell(u) = u^\ell = D_{0,n}(u) = P^n(u)$ $\square$

Lemma 9.8 $u \in \tilde{H}^{p,q}$ $n > p-q$, $n \geq q \Rightarrow P^n(u) = 0$

(Axiom 4)

$i = n+q-p$ $\qquad j = n-q$

$$\begin{tikzcd}
\sigma_s^i \sigma_t^j(u) \in \tilde{H}^{2n,n}(\Sigma_s^i \Sigma_t^j \mathcal{F}_\bullet, \mathbb{Z}/\ell) \arrow[d, mapsto, "\text{Lemma 9.7}"'] \arrow[d, "P^n"] \arrow[r, phantom, "\cong"] & \tilde{H}^{p,q}(\mathcal{F}_\bullet, \mathbb{Z}/\ell) \ni u \arrow[d, "P^n"'] \arrow[d, mapsto] \\
(\sigma_s^i \sigma_t^j(u))^\ell \in \tilde{H}^{2n+2n(\ell-1), n+n(\ell-1)} \arrow[r, phantom, "\cong"] & \tilde{H}^{p+2i(\ell-1), q+i(\ell-1)} \ni P^n(u)
\end{tikzcd}$$

$= 0$ because cup product of simplicial suspension is trivial

$\square$

Total power operation

$$R : \tilde{H}^{*,*} \to \tilde{H}^{*,*}[\![c, d^{\pm 1}]\!] / c^2 = \tau d + \rho c$$

$$\tau = \rho = 0 \quad \ell \neq 2$$

$$R(u) := \sum_i B^{i-1}(u)\, c d^{-i} + P^i(u)\, d^{-i}$$

For $\ell = 2$ this becomes

$$R(u) = \sum Sq^{2i-1}(u)\, c d^i + Sq^{2i}(u)\, d^i$$

Claim: $R(uv) = R(u) \cdot R(v)$

Pf: $u \in \tilde{H}^{2n,n}$, $v \in \tilde{H}^{2m,m}$

$$\begin{aligned} d^n \cdot R(u) &= \sum_i B^{i-1}(u)\, c d^{n-i} + P^i(u)\, d^{n-i} \\ &= \sum_i C_{n-i+1}(u)\, c d^{n-i} + D_{n-i}(u)\, d^{n-i} \\ &= \sum_j C_{j+1}(u)\, c d^j + D_j(u)\, d^j \\ &= P_\ell(u) \end{aligned}$$

Lemma 5.7

$$\begin{array}{ccc} P_\ell(u \wedge v) & = & \Delta^*(P_\ell(u) \wedge P_\ell(v)) \\ \| & & \| \\ d^{n+m} R(u \wedge v) & & \Delta^*(d^n R(u) \wedge d^m R(v)) \\ & & = d^{n+m} R(u) R(v) \end{array}$$

$\square$